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THE PLANE VERTICAL WAVEMAKER PROBLEM - REVISITED

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Abstract. The classical two-dimensional plane vertical wavemaker problem of Havelock is reinvestigated by using the Fourier cosine integral transform in the horizontal co-ordinate. Exploiting a property of this transform, the amplitude (complex) of the wave at large distances from the wavemaker is obtained in a natural manner.

1. INTRODUCTION

The problem of forced surface waves produced by a plane vertical wavemaker was treated by Havelock [1] long back within the framework of linearised theory of water waves. Assuming irrotational motion, the form of the velocity potential (and hence the free surface elevation) was given in [1] by a suitable combination of free progressive wave solution and elementary solutions which die out at large distances from the wavemaker in terms of integrals involving the prescribed normal fluid velocity on the wavemaker. In the present note we demonstrate the use of the powerful technique of integral transform to solve the two-dimensional plane vertical wavemaker problem.

2. FORMULATION OF THE PROBLEM

A plane vertical time-harmonic wavemaker is immersed in an inviscid homogeneous fluid. We use a rectangluar cartesian coordinate system in which the y-axis is taken vertically downwards, the plane $y = 0$ is the undisturbed position of the free surface, the wavemaker is the plane $x = 0$ and in the undisturbed state the fluid occupies the region $y \geq 0$, $|x| > 0$ assuming it to be of infinite depth. The motion in the fluid is under gravity and is due to prescribed normal fluid velocity $Re\,\{U(y)e^{-i\sigma t}\}$ at the wavemaker where σ is the circular frequency, t represents time and $U(y) \to 0$ as $y \to \infty$. Assuming the motion in the fluid to be irrotational, it can be described by a velocity potential $\varphi(x, y, t)$ which is the real part of $\Psi(x, y)e^{-i\sigma t}$. Then $\Psi(x, y)$ satisfies

$$\nabla^2\Psi = 0 \text{ in the fluid region,} \tag{2.1}$$

the linearised free surface condition

$$K\Psi + \Psi_y = 0 \text{ on } y = 0, |x| > 0, \tag{2.2}$$

where $K = \sigma^2/g$, g being the gravity, the wavemaker condition

$$\Psi_x = U(y) \text{ on } x = 0, \tag{2.3}$$

the bottom condition

$$\nabla\Psi \to 0 \text{ as } y \to \infty \text{ for fluid of infinite depth,} \tag{2.4}$$

and the condition that it represents outgoing waves at large distances from the wavemaker, which can be mathematically expressed as

$$\Psi \to Ce^{\{-Ky+iK\,|x|\}} \text{ as } |x| \to \infty \tag{2.5}$$

We may note that C in (2.5) is unknown and is to be determined. This will be determined by exploiting a property of the Fourier cosine transform to be used in the next section. It is sufficient to solve the problem in the region $x > 0$, because of the symmetry about $x = 0$

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SOLUTION OF THE PROBLEM

Let $\Psi(x, y) = Ce^{\{-Ky+iKx\}} + \Phi(x, y)$
thenΦ satisfies

$$\left.\begin{aligned} \nabla^2\Phi = 0 \quad 0 < y < \infty, x > 0 \\ K\Phi + \Phi y = 0 \quad \text{on } y = 0, x > 0 \\ \nabla\Phi \to 0 \quad \text{as } y \to \infty \\ \Phi \to 0 \quad \text{as } x \to \infty \\ \Phi_x = V(y) \quad \text{on } x = 0 \end{aligned}\right\} \tag{3.1}$$

where

$$V(y) = U(y) - CiKe^{-Ky}. \tag{3.2}$$

Let

$$\chi(\xi, y) = \int_0^\infty \Phi(x, y) \cos \xi x \, dx, \tag{3.3}$$

then χ satisfies

$$\left.\begin{aligned} &\chi_{yy} - \xi^2\chi = V(y), \ 0 < \ y \ < \ \infty, \\ &K\chi + \chi_y = 0 \text{ on } y = 0, \\ &\chi_y \to 0 \text{ as } y \to \infty. \end{aligned}\right\} \tag{3.4}$$

The solution of (3.4) is given by (cf. Mikhlin [3])

$$\chi(\xi, y) = -\int_0^\infty G(y, s) \, V(s) \, ds \tag{3.5}$$

where $G(y, s)$ is the associated Green's function defined by

$$G(y, s) = \frac{(\xi \cosh \xi y - K \sinh \xi y)e^{-\xi s}}{\xi(\xi - K)} \quad \text{for } 0 \le y \le s. \tag{3.6}$$

For $0 \le s \le y$, y and s in (3.6) are to be interchanged.
Using (3.6) and (3.2) in (3.5) we obtain

$$\chi(\xi, y) = \frac{\xi \cosh \xi y - K \sinh \xi y}{\xi} \left[\frac{iKC}{\xi^2 - K^2} - \frac{a(\xi)}{\xi - K}\right] \tag{3.7}$$

where

$$a(\xi) = \int_0^\infty U(s)e^{-\xi s} \, ds. \tag{3.8}$$

Now $\chi(\xi, y)$ is the Fourier cosine transform of a certain function, and treated as a function of the complex variable ξ, it cannot have a singularity on the real axis. This immediately suggests that the unknown constant C appearing in (3.7) must be such that $\chi(\xi, y)$ has no singulatiry at the point $\xi = K$. This gives

$$C = -2ia(K). \tag{3.9}$$

Hence we obtain

$$\chi(\xi, y) = \frac{\xi \cosh \xi y - K \sinh \xi y}{\xi(\xi - K)} \left\{ \frac{2K}{\xi + K} a(K) - a(\xi) \right\}$$

so that

$$\Phi(x, y) = \frac{2}{\pi} \int_0^\infty \frac{\xi \cosh \xi y - K \sinh \xi y}{\xi(\xi - K)} \left\{ \frac{2K}{\xi + K} a(K) - a(\xi) \right\} \cos \xi x \, d\xi, \ x > 0 \tag{3.10}$$

where the integrals is in the sense of Cauchy principal value. Now writing $2\cos \xi x$ as $e^{i\xi x} + e^{-i\xi x}$ and rotating the contour along the positive imaginary axis for the integral involving $e^{i\xi x}$ and along the negative imaginary axis for the integral involving $e^{-i\xi x}$ we obtain

$$\begin{aligned} \Phi(x, y) &= -\frac{1}{\pi} \int_0^\infty \frac{k \cos ky - K \sin ky}{k(k^2 + K^2)} e^{-kx} [k \{a(ik) + a(-ik)\} \\ &\quad - iK \{a(ik) - a(-ik)\}] \, dk \\ &= -\frac{2}{\pi} \int_0^\infty \int_0^\infty \frac{e^{-kx}(k \cos ky - K \sin ky)(k \cos ks - K \sin ks)}{k(k^2 + K^2)} dk \ U(s) ds \end{aligned} \tag{3.11}$$

where we have employed (3.8). Hence $\Psi(x, y)$ is found and we finally obtain

$$\begin{aligned} \varphi(x, y) &= Re \left\{ \Psi(x, y) e^{-i\sigma t} \right\} = -2e^{-Ky} \sin(\sigma t - Kx) \int_0^\infty U(s) e^{-Ks} ds \\ &\quad - \frac{2}{\pi} \cos \sigma t \int_0^\infty \int_0^\infty \frac{e^{-kx}(k \cos ky - K \sin ky)(k \cos ks - K \sin ks)}{k(k^2 + K^2)} dk U(s) ds, \quad x > 0 \end{aligned} \tag{3.12}$$

This coincides with the result given in [1]. For $x < 0, x$ in (3.12) is to be replaced by $-x$.

4. Discussion

The integral transform method demonstrated above avoids the use of the complicated technique involving the Green's integral theorem in the fluid domain
(cf. Rhodes-Robinson [2]). A similar method is also applicable when the fluid is of uniform finite depth and/or the effect of surface tension at the free surface is taken into consideration.

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